Applied Mathematics Letters 22 (2009) 1248–1251

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

On the exact solutions for initial value problems of second-order differential equations

Lazhar Bougoffa

Al-imam University, Faculty of Science, Department of Mathematics, P.O.Box 90950, Riyadh 11623, Saudi Arabia

ARTICLE INFO

Article history:
Received 24 April 2008
Received in revised form 15 December 2008
Accepted 26 January 2009

Keywords:
Second-order differential equation
Exact solutions
Inverse differential operator

ABSTRACT

In this paper, the solutions of initial value problems for a class of second-order linear differential equations are obtained in the exact form by writing the equations in the general operator form and finding an inverse differential operator for this general operator form.

1. Introduction

The consideration of initial value problems for second-order ordinary differential equations is motivated by a number of physical problems in various fields [1,2].

In recent years, the studies of these types of initial value problems have attracted the attention of many mathematicians and physicists. For example, Adomian's decomposition method (ADM) [3,4] which has been applied to a wide class of initial and boundary value problems for differential equations.

The solution proposed by Adomian [3,4] is to take the differential operator L as the highest-ordered derivative of the linear part. For example, for the linear (deterministic) ordinary differential equation [4]

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - kx^p u = f(x) \quad \text{with } u(-1) = u(1) = 0.$$

Adomian rewrites this equation in the operator form $L_{xx}u = F(u)$, where $L_{xx} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $F(u) = kx^p u + f$ and defines L_{xx}^{-1} as $L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)\mathrm{d}x\mathrm{d}x$, and operates with L_{xx}^{-1}. Therefore

$$u = u(0) + xu'(0) + L_{xx}^{-1}\left(kx^p u\right) + L_{xx}^{-1}\left(f(x)\right),$$

and the ADM consists of representing the solution u in the decomposition form given by $u = \sum_{n=0}^{\infty} u_n$.

Many solutions have been obtained in [5,6] for further specific second-order ordinary differential equations: Lane–Emden equation, linear singular initial value problem and other equations by choosing a different inverse differential operators and using ADM.

In the present paper, we give a novel approach for obtaining the exact solutions of the following initial value problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)\frac{\mathrm{d}y}{\mathrm{d}x} + r(x)y = f(x), \quad x > x_0, \tag{1.1}$$

$$y(x_0) = \alpha, \qquad y'(x_0) = \beta, \tag{1.2}$$

where $p(x) \in C^1([x_0, L])$, $q(x)$, $r(x)$ and $f(x)$ are some functions.

E-mail address: bougoffa@hotmail.com.

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.01.038

The method is based on writing Eq. (1.1), under suitable conditions on the coefficients, in the general operator form $L_{xx}z = g(x)$, where $L_{xx}z \equiv \frac{d}{dx}\left(h(x)\frac{dz}{dx}\right)$ and we propose an inverse differential operator L_{xx}^{-1} of L_{xx}. Therefore, the exact solutions of the problem (1.1)–(1.2) can be obtained from operating with L_{xx}^{-1}.

2. The method

The key idea of our method is as follows.

Multiplying both sides of Eq. (1.1) by $\xi_1(x) = e^{\int \frac{r(x)}{q(x)}dx}$, we get

$$\xi_1(x)\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \xi_1(x)q(x)\frac{dy}{dx} + \xi_1(x)r(x)y = \xi_1(x)f(x),$$

taking into account $\xi_1'(x)q(x) = \xi_1(x)r(x)$, we obtain

$$\xi_1(x)\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \xi_1(x)q(x)\frac{dy}{dx} + \xi_1'(x)q(x)y = \xi_1(x)f(x), \tag{2.1}$$

$$\xi_1(x)\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\frac{d}{dx}(\xi_1(x)y) = \xi_1(x)f(x), \tag{2.2}$$

so that

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \frac{q(x)}{\xi_1(x)}\frac{d}{dx}(\xi_1(x)y) = f(x). \tag{2.3}$$

Let $\xi_1(x)y = z$, where $\xi_1(x_0)$ and $\xi_1'(x_0)$ are defined.

Substituting this into Eq. (2.3), we get

$$\frac{d}{dx}\left(p(x)\left(\frac{1}{\xi_1(x)}\right)'z + \frac{p(x)}{\xi_1(x)}\frac{dz}{dx}\right) + \frac{q(x)}{\xi_1(x)}\frac{dz}{dx} = f(x). \tag{2.4}$$

Hence Eq. (2.4) may be rewritten as

$$\frac{d}{dx}\left(\frac{p(x)}{\xi_1(x)}\frac{dz}{dx}\right) + \frac{d}{dx}\left(p(x)\left(\frac{1}{\xi_1(x)}\right)'z\right) + \frac{q(x)}{\xi_1(x)}\frac{dz}{dx} = f(x). \tag{2.5}$$

If we choose $p(x)(\frac{1}{\xi_1(x)})' = c$, where c is a constant, that is, $cq(x) + p(x)r(x)e^{-\int \frac{r(x)}{q(x)}dx} = 0$. Then, with this choice, Eq. (2.5) becomes

$$\frac{d}{dx}\left(s(x)\frac{dz}{dx}\right) + t(x)\frac{dz}{dx} = f(x), \tag{2.6}$$

where $s(x) = \frac{p(x)}{\xi_1(x)}$ and $t(x) = \frac{q(x)}{\xi_1(x)} + c$.

Now, as before, multiplying both sides of Eq. (2.6) by $\xi_2(x) = e^{\int \frac{t(x)}{s(x)}dx}$, we get

$$\xi_2(x)\frac{d}{dx}\left(s(x)\frac{dz}{dx}\right) + \xi_2(x)t(x)\frac{dz}{dx} = \xi_2(x)f(x). \tag{2.7}$$

Taking into account $\xi_2'(x)s(x) = \xi_2(x)t(x)$, we obtain

$$\xi_2(x)\frac{d}{dx}\left(s(x)\frac{dz}{dx}\right) + \xi_2'(x)s(x)\frac{dz}{dx} = \xi_2(x)f(x), \tag{2.8}$$

so that

$$\frac{d}{dx}\left(\xi_2(x)s(x)\frac{dz}{dx}\right) = \xi_2(x)f(x). \tag{2.9}$$

Now, we write Eq. (2.9) in the form

$$L_{xx}z = g(x), \tag{2.10}$$

where $L_{xx}z \equiv \frac{d}{dx}\left(h(x)\frac{dz}{dx}\right)$, $h(x) = \xi_2(x)s(x)$ and $g(x) = \xi_2(x)f(x)$.

A formal inverse of (2.10) can be easily found. We choose it as

$$L_{xx}^{-1}z(x)=\int_{x_0}^{x}\frac{\mathrm{d}t}{h(t)}\int_{x_0}^{t}z(s)\mathrm{d}s,$$

where $L_{xx}^{-1}L_{xx}\neq L_{xx}L_{xx}^{-1}$. Applying L_{xx}^{-1} to Eq. (2.10), we see that

$$(L_{xx}^{-1}L_{xx})z(x)=\int_{x_0}^{x}\frac{\mathrm{d}t}{h(t)}\int_{x_0}^{t}\left(h(s)z'(s)\right)'\mathrm{d}s,$$
$$(L_{xx}^{-1}L_{xx})z(x)=\int_{x_0}^{x}\left(h(t)z'(t)-h(x_0)z'(x_0)\right)\frac{\mathrm{d}t}{h(t)},$$

so that

$$(L_{xx}^{-1}L_{xx})z(x)=z(x)-z(x_0)-h(x_0)z'(x_0)\int_{x_0}^{x}\frac{\mathrm{d}t}{h(t)}.$$

Therefore, we obtain

$$z(x)=z(x_0)+h(x_0)z'(x_0)\int_{x_0}^{x}\frac{\mathrm{d}t}{h(t)}+L_{xx}^{-1}(g(x)). \tag{2.11}$$

After z has been found the solution of (1.1)–(1.2) is given by $y=\frac{1}{\xi_1(x)}z$.

Thus, we have proved the following new theorem.

Theorem 1. *For the given initial value problem (1.1)–(1.2).*

If there exists a constant c such that

$$cq(x)+p(x)r(x)\mathrm{e}^{-\int\frac{r(x)}{q(x)}\mathrm{d}x}=0.$$

Then, the solution is given by

$$y=\frac{1}{\xi_1(x)}z,$$

where

$$z(x)=z(x_0)+h(x_0)z'(x_0)\int_{x_0}^{x}\frac{\mathrm{d}t}{h(t)}+L_{xx}^{-1}(g(x)),$$
$$h(x)=\xi_2(x)s(x),\qquad g(x)=\xi_2(x)f(x),$$
$$\xi_1(x)=\mathrm{e}^{\int\frac{r(x)}{q(x)}\mathrm{d}x},\qquad \xi_2(x)=\mathrm{e}^{\int\frac{t(x)}{s(x)}\mathrm{d}x},$$
$$s(x)=\frac{p(x)}{\xi_1(x)},\qquad t(x)=\frac{q(x)}{\xi_1(x)}+c,$$
$$z(x_0)=\alpha\xi_1(x_0),\qquad z'(x_0)=\alpha\xi_1'(x_0)+\beta\xi_1(x_0)$$

and

$$L_{xx}^{-1}g(x)=\int_{x_0}^{x}\frac{\mathrm{d}t}{h(t)}\int_{x_0}^{t}g(s)\mathrm{d}s.$$

In the following we shall apply the above techniques to a few various linear differential equations of mathematical physics.

Example 1 (*Degenerate Hypergeometric Equation*). Consider the singular initial value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{b-x}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{a}{x}y=0,\quad x>0,$$
$$y(0)=1,\qquad y'(0)=\frac{-1}{b}.$$

Here $p(x)=1$, $q(x)=\frac{b-x}{x}$, $r(x)=\frac{b}{x}$ and $f(x)=0$.

If we choose $a=-1$, then, the conditions of Theorem 1 are fulfilled and straightforward computation yields $c=1$, $\xi_1(x)=\frac{1}{b-x}$, $\xi_2(x)=\frac{x^b}{b-x}\mathrm{e}^{-x}$, $h(x)=x^b\mathrm{e}^{-x}$, $s(x)=b-x$ and $t(x)=\frac{(b-x)^2}{x}+1$.

By direct application of Theorem 1, we get $z=\frac{1}{b}$ and the exact solution to this problem $y(x)=\frac{1}{\xi_1(x)}z=\frac{b-x}{b}$.

Example 2 (*Euler Equation*). Consider the initial value problem

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=3x^2,\quad x>1,$$
$$y(1)=1,\qquad y'(1)=2.$$

If we choose $a = 1$ and $b = -1$, then the conditions of Theorem 1 are fulfilled and direct calculation produces $c = 1$, $\xi_1(x)=\frac{1}{x}$, $\xi_2(x)=x^2$, $h(x)=x^3$, $t(x)=2$, $s(x)=x$, $g(x)=3x^2$ and $L_{xx}^{-1}g(x)=\frac{1}{2x^2}+x-\frac{3}{2}$.

By direct application of Theorem 1, we get $z = x$. Therefore, the exact solution to this problem is $y(x)=\frac{1}{\xi_1(x)}z=x^2$.

Example 3 (*Legendre Equation*). Consider the initial value problem

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{2}{1-x^2}y=\frac{2}{1-x^2},\quad x>-1,$$
$$y(-1)=2,\qquad y'(-1)=-1.$$

Theorem 1 can be applied and direct calculation produces $c=1,\ \xi_1(x)=\frac{1}{x},\ \xi_2(x)=x(1-x^2),\ h(x)=x^2(1-x^2),\ t(x)=\frac{-2x}{1-x^2}+1,\ s(x)=x,\ g(x)=2x$ and $L_{xx}^{-1}g(x)=\frac{1}{x}+1$. Thus $z=\frac{1}{x}-1$ and $y(x)=1-x$.

3. Conclusion

In conclusion, we have successfully found some exact solutions for a second-order ordinary differential equations by using a direct method. The idea of this method is to change the problem for solving (1.1) to the general operator form $L_{xx}z\equiv\frac{d}{dx}\left(h(x)\frac{dz}{dx}\right)$ in which the inverse differential operator L_{xx}^{-1} of L_{xx} can be found. Therefore, the exact solutions of such problem (1.1)–(1.2) are obtained from operating with L_{xx}^{-1}.

References

[1] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover Publications, New York, 1962.
[2] E. Groswald, Bessel Polynomials, Springer, Berlin, 1978.
[3] G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, Orlando, FL, 1986.
[4] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1994.
[5] M.M. Hosseini, H. Nasabzadeh, Modified Adomian decomposition method for specific second order ordinary differential equations, Appl. Math. Comput. 186 (2007) 117–123.
[6] A.M. Wazwaz, A new method for solving singular initial value problems in the second-order ordinary differential equations, Appl. Math. Comput. 128 (2002) 45–57.